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Propagation of disturbances in a magnetized relativistic plasma of very low density

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Abstract. Propagation of disturbances through relativistic Vlasov plasmas in the presence of a uniform external magnetic field has been investigated. The main purpose of this paper is to solve the dispersion relation in exact expressions for an arbitrary isotropic equilibrium distribution function. Analytic expressions in the form of single integrals have been derived for the propagations parallel and perpendicular to the external magnetic field. Expressions for the limits and asymptotic values of the frequency ratios have also been obtained. Numerical results for Maxwellian distribution are depicted graphically.

1. Introduction

Plasmas of very high temperature exist in fusion reactors. In order that two nuclei may fuse together in the presence of strong repulsive forces, temperatures higher than 10^8 K are required. At these temperatures, the electrons attain a thermal speed equivalent to one-fifth of the speed of light. The density of thermonuclear plasmas is very low, around 10^{16} particles per cm^3. Thus, their behaviour can be described by the relativistic Boltzmann–Vlasov equation in association with the self-consistent field equations.

Wave propagation through a relativistic Vlasov plasma in the absence of external magnetic field has been investigated by numerous authors (Clemmow and Willson 1956, Kursunoglu 1961, Buti 1963a, Misra 1975). Buti (1963b) also extended his analysis to the case with a uniform magnetic field. He derived the dispersion relations in the extreme relativistic limit. For wave propagation parallel to the magnetic field, a Maxwellian distribution was used; and the dispersion relations in the limits of weak and strong magnetic fields were derived. For wave propagation normal to the magnetic field, an isotropic non-Maxwellian distribution was employed; and the uncoupled transverse mode in the extreme relativistic limit for the strong magnetic field was studied.

In the present paper, we consider the relativistic electron plasma of very low density moving over a uniform background of immobile protons in the presence of an external magnetic field. Our purpose is to solve the dispersion relations in exact analytic expressions for an arbitrary isotropic equilibrium distribution. The limits of frequencies and the effects and limits of external magnetic field for coupled and uncoupled modes are investigated in detail.

2. Dispersion relation

By applying Fourier transformation to the relativistic Boltzmann–Vlasov equation and Maxwell's equations, Buti (1963a) has derived a matrix equation for the electric field

$$\boldsymbol{A} \cdot \tilde{\boldsymbol{E}} \equiv \frac{c^2}{\mu\epsilon}\frac{\boldsymbol{k}}{\omega}\times\left(\frac{\boldsymbol{k}}{\omega}\times\tilde{\boldsymbol{E}}\right)+\left(\boldsymbol{I}-2\pi\frac{\omega_{\mathrm{p}0}^2}{\omega\omega_{\mathrm{b}}}\boldsymbol{R}\right)\cdot\tilde{\boldsymbol{E}}=0 \tag{2.1}$$

where $\tilde{\boldsymbol{E}}$ denotes the Fourier transform of electric field, $\boldsymbol{A}$ the square matrix defined by equation (2.1) for mathematical simplicity, $\boldsymbol{I}$ the unit matrix, c the speed of light, μ the magnetic permeability, ϵ the electric permittivity, ω the wave frequency, $\boldsymbol{k}$ the propagation vector, $\omega_{\mathrm{p}0}$ and ω_{b} the parameters defined by

$$\omega_{\mathrm{p}0}^2 = 4\pi n_0 q^2/(m_0\epsilon), \qquad \omega_{\mathrm{b}} = qB_0/(m_0 c),$$

n_0 the equilibrium number density, q and m_0 the electric charge (positive value) and the rest mass of an electron, and B_0 the external magnetic field. Without loss of generality, the vector $\boldsymbol{B}_0$ may be set in the z-direction and the vector $\boldsymbol{k}$ in the xz-plane, while the angle between these two vectors is denoted by θ. In addition, the proper velocity $\boldsymbol{u}$ is specified in spherical coordinates by (u, θ', ϕ') and the vector $\tilde{\boldsymbol{E}}$ by $(\tilde{E}, \theta'', \phi'')$. Then the matrix $\boldsymbol{R}$ may be expressed as

$$\boldsymbol{R}(\boldsymbol{k}, \omega) = \frac{\omega_{\mathrm{p}0}^2}{\omega\omega_{\mathrm{b}}}\int_0^\infty \mathrm{d}u\, u^3\frac{\mathrm{d}f_0}{\mathrm{d}u}\int_0^\pi \mathrm{d}\theta' \sin\theta' \int_0^{2\pi}\mathrm{d}\phi'\, \boldsymbol{e}_u(\phi')\boldsymbol{F}(\phi'), \tag{2.2}$$

$$\boldsymbol{F}(\phi') = \mathrm{i}\exp[\mathrm{i}(\lambda\phi' - \gamma\sin\phi')]\int^{\phi'} \boldsymbol{e}_u(\chi)\exp[\mathrm{i}(\gamma\sin\chi - \lambda\chi)]\,\mathrm{d}\chi, \tag{2.3}$$

where f_0 denotes the equilibrium distribution function, $\boldsymbol{e}_u$ the unit vector in the direction of $\boldsymbol{u}$, and γ and λ the dimensionless parameters defined as

$$\gamma = (ku/\omega_{\mathrm{b}})\sin\theta\sin\theta'$$

$$\lambda = \frac{\omega}{\omega_{\mathrm{b}}}\left(1+\frac{u^2}{c^2}\right)^{1/2} - \frac{ku}{\omega_{\mathrm{b}}}\cos\theta\cos\theta'.$$

To ensure finite values of $\tilde{\boldsymbol{E}}$, we obtain the dispersion relation

$$\det \boldsymbol{A} = 0.$$

Equation (2.1) indicates that solutions of the dispersion relation rely essentially on the possible evaluation of $\boldsymbol{R}$. The integration in expression (2.3) may be performed analytically by expanding $\exp(\mathrm{i}\gamma\sin\phi')$ in terms of Bessel functions of the first kind (Pierpont 1959 p 555):

$$\cos(\gamma\sin\phi') = J_0(\gamma) + 2\sum_{n=1}^{\infty} J_{2n}(\gamma)\cos(2n\phi') \tag{2.4}$$

$$\sin(\gamma\sin\phi') = 2\sum_{n=1}^{\infty} J_{2n-1}(\gamma)\sin[(2n-1)\phi']. \tag{2.5}$$

Subject to the periodic condition, $\boldsymbol{F}(\phi')$ may be expressed as

$$\boldsymbol{F}(\phi')=\sum_{n=-\infty}^{\infty} J_n(\gamma)\exp[\mathrm{i}(n\phi'-\gamma\sin\phi')]\left[\frac{1}{2}\left(\frac{\boldsymbol{e}_x-\mathrm{i}\boldsymbol{e}_y}{n-\lambda+1}\exp(\mathrm{i}\phi')\right.\right.$$
$$\left.\left.+\frac{\boldsymbol{e}_x+\mathrm{i}\boldsymbol{e}_y}{n-\lambda-1}\exp(-\mathrm{i}\phi')\right)+\frac{\boldsymbol{e}_z}{n-\lambda}\right] \tag{2.6}$$

where $\boldsymbol{e}$ denotes the unit vector.

With the aid of expressions (2.4) and (2.5), the following definite integrals for n as an integer or zero can be derived:

$$\int_0^{2\pi}\exp[\mathrm{i}(n\phi'-\gamma\sin\phi')]\,\mathrm{d}\phi'=2\pi J_n(\gamma) \tag{2.7}$$

$$\int_0^{2\pi}\exp[\mathrm{i}(n\phi'-\gamma\sin\phi')]\cos\phi'\,\mathrm{d}\phi'=2\pi\frac{n}{\gamma}J_n(\gamma). \tag{2.8}$$

$$\int_0^{2\pi}\exp[\mathrm{i}(n\phi'-\gamma\sin\phi')]\sin\phi'\,\mathrm{d}\phi'=2\pi\mathrm{i}J_n'(\gamma). \tag{2.9}$$

Substituting expression (2.6) into (2.1) and using formulae (2.7)–(2.9), the expression for $\boldsymbol{R}$ may first be integrated with respect to ϕ' over 2π. If we introduce a sequence of vectors $\boldsymbol{U}_n$ defined as

$$\boldsymbol{U}_n=\left(\frac{n\omega_b}{k\sin\theta}\boldsymbol{e}_x+u\cos\theta'\boldsymbol{e}_z\right)J_n(\gamma)+\mathrm{i}u\sin\theta' J_n'(\gamma)\boldsymbol{e}_y \tag{2.10}$$

then the dyadic $\boldsymbol{R}$ may be expressed as

$$\boldsymbol{R}(\boldsymbol{k},\omega)=\int_0^{\infty}\mathrm{d}u\,u\frac{\mathrm{d}f_0}{\mathrm{d}u}\int_0^{\pi}\mathrm{d}\theta'\sin\theta'\sum_{n=-\infty}^{\infty}\frac{\boldsymbol{U}_n\boldsymbol{U}_n^*}{n-\lambda} \tag{2.11}$$

where the asterisk indicates the complex conjugate. Let $\boldsymbol{S}$ denote the infinite series

$$\boldsymbol{S}=\sum_{n=-\infty}^{\infty}(n-\lambda)^{-1}\boldsymbol{U}_n\boldsymbol{U}_n^*. \tag{2.12}$$

Physically $\boldsymbol{S}/2\pi$ may be interpreted as the density of $\boldsymbol{R}$ in the velocity space.

Before we actually evaluate the expressions for $\boldsymbol{R}$, we first state here some fundamental properties of $\boldsymbol{R}$:

(1) Expression (2.10) indicates that the dyadic $\boldsymbol{U}_n\boldsymbol{U}_n^*$ is Hermitian. Among the off-diagonal elements $(U_nU_n^*)_{xz}$ is real while $(U_nU_n^*)_{xy}$ and $(U_nU_n^*)_{yz}$ are both imaginary. Thus

$$R_{yx}=-R_{xy},\qquad R_{zy}=-R_{yz},\qquad R_{zx}=R_{xz}.$$

(2) For wave propagation parallel to $\boldsymbol{B}_0$, γ is zero. By expressions (2.10) and (2.12),

the elements of $\boldsymbol{S}$ may be expressed as

$$S_{xx}=S_{yy}=\frac{u^2}{2}\frac{\lambda}{1-\lambda^2}\sin^2\theta',$$
$$S_{zz}=-\frac{u^2}{\lambda}\cos^2\theta', \tag{2.13}$$
$$S_{yx}=-S_{xy}=\frac{\mathrm{i}}{2}\frac{u^2}{1-\lambda^2}\sin^2\theta'.$$

All other elements are zero.

(3) For wave propagation normal to $\boldsymbol{B}_0$, γ is an even function of $\cos\theta'$. Expression (2.10) indicates that both S_{xz} and S_{yz} are odd functions of $\cos\theta'$. Although in general none of the elements of $\boldsymbol{S}$ is zero, after integration with respect to θ' both R_{xz} and R_{yz} vanish.

(4) Consider the waves propagating in two symmetric directions θ_1 and $\theta_2=\pi-\theta_1$. For these two directions, γ and $\boldsymbol{U}_n$ are the same, while λ and $\boldsymbol{S}$ are different. However, by integration with respect to θ' over π, the dyadic $\boldsymbol{R}$ remains the same.

(5) When $\boldsymbol{B}_0$ is absent, both ω_b and $\boldsymbol{R}$ become zero, but their ratio has a finite value. Now since $\boldsymbol{B}_0$ does not exist, there is no reason to keep $\boldsymbol{k}$ away from the z-axis. Thus, we rotate the coordinate system about the y-axis by an angle of θ, and define

$$\Omega=\omega_\mathrm{b}\lambda=\omega(1+u^2/c^2)^{1/2}-\boldsymbol{k}\cdot\boldsymbol{u}.$$

In this limiting case, the dyadic $\boldsymbol{R}$ is diagonal. Equivalently,

$$\lim_{\boldsymbol{B}_0\to 0}\frac{\boldsymbol{S}}{\omega_\mathrm{b}}=-\frac{u^2}{\Omega}\left(\frac{\sin^2\theta'}{2}(\boldsymbol{e}_{xx}+\boldsymbol{e}_{yy})+\cos^2\theta'\boldsymbol{e}_{zz}\right).$$

3. Propagation parallel to $\boldsymbol{B}_0$

When $\theta=0$ or π, A_{xz} and A_{yz} vanish. Thus the dispersion relation is reduced to

$$2\pi\frac{\omega_{\mathrm{p}0}^2}{\omega\omega_\mathrm{b}}R_{zz}=1 \tag{3.1}$$

and

$$2\pi\frac{\omega_{\mathrm{p}0}^2}{\omega\omega_\mathrm{b}}(R_{xx}\pm\mathrm{i}R_{xy})=1-\frac{c^2}{\mu\epsilon}\left(\frac{k}{\omega}\right)^2. \tag{3.2}$$

Relation (3.1) describes the propagation of longitudinal waves, while relations (3.2) represent the coupled transverse waves.

3.1. Longitudinal mode

The solution of equation (3.1) derived from expressions (2.11) and (2.13), which is independent of $\boldsymbol{B}_0$, may be expressed as

$$\left(\frac{k}{\omega_{\mathrm{p}0}}\right)^2=2\left[Z_\parallel\left(\frac{\omega}{k}\right)-M\right], \tag{3.3}$$

where the function $Z_{\|}$ is defined as

$$Z_{\|}(x) = -2\pi x \int_0^{\infty} \frac{\mathrm{d}f_0}{\mathrm{d}u}\left(1+\frac{u^2}{c^2}\right) \tanh^{-1}\left[\frac{u}{x}\left(1+\frac{u^2}{c^2}\right)^{-1/2}\right] \mathrm{d}u, \tag{3.4}$$

and the constant M is defined as

$$M = -2\pi \int_0^{\infty} \frac{\mathrm{d}f_0}{\mathrm{d}u} u\left(1+\frac{u^2}{c^2}\right)^{1/2} \mathrm{d}u. \tag{3.5}$$

From equation (3.3) together with expressions (3.4) and (3.5), we obtain the frequency limits

$$\left(\frac{\omega}{\omega_{\mathrm{p}0}}\right)^2 = -4\pi c \int_0^{\infty} f_0 u \left(2 \sinh^{-1}\frac{u}{c} - u(c^2+u^2)^{-1/2}\right) \mathrm{d}u \tag{3.6}$$

for $\omega/k = c$, and

$$\left(\frac{\omega}{\omega_{\mathrm{p}0}}\right)^2 = -\frac{4\pi}{3} \int_0^{\infty} \frac{\mathrm{d}f_0}{\mathrm{d}u} u^3\left(1+\frac{u^2}{c^2}\right)^{-1/2} \mathrm{d}u \tag{3.7}$$

for $\omega/k = \infty$. Let R_{∞} denote the value given by expression (3.7) for future reference. In expression (3.4) the argument of $\tanh^{-1}$ must not be greater than unity; otherwise the function $Z_{\|}$ becomes infinite. This confirms the fact that no wave can propagate at sub-relativistic speeds.

Figure 1 shows the variation of frequency ratio $\omega/\omega_{\mathrm{p}0}$ in the limit of real frequency. Note that although all the equations derived in this paper are valid for any isotropic f_0, the numerical illustrations are made for the Maxwellian distribution

$$f_0 = \beta \exp\left[-\frac{m_0 c^2}{\kappa T}\left(1+\frac{u^2}{c^2}\right)^{1/2}\right] \tag{3.8}$$

with $m_0 c^2/(\kappa T) = 50$ (corresponding to a temperature of $T = 11{\cdot}86$ K or a thermal speed of $0{\cdot}2\,c$), where β denotes the normalization coefficient, and κ Boltzmann's constant. Numerical results indicate that the frequency ratio $\omega/\omega_{\mathrm{p}0}$ varies from 1·005 at $\omega/k = c$ to 0·976 at $\omega/k = \infty$. It reveals a relativistic effect up to 2·4% at the temperature considered, noting that $\omega = \omega_{\mathrm{p}0}$ in the non-relativistic case.

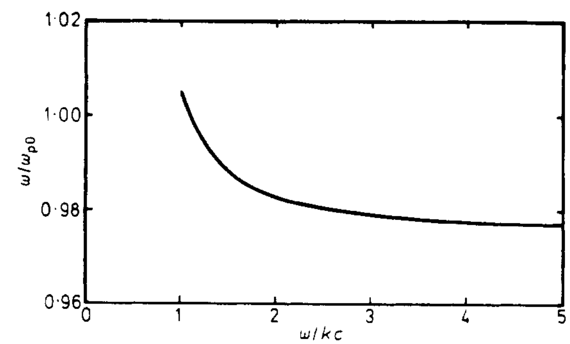

Figure 1. Longitudinal oscillations propagating in a direction parallel to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0 c^2/(\kappa T) = 50$.

3.2. Coupled transverse modes

Similarly, by expressions (2.11) and (2.13), equations (3.2) yield the relations

$$W\left(\frac{\omega}{k}, \pm\frac{\omega_b}{\omega}\right) \pm K\frac{\omega_b}{\omega} = M - \left(\frac{k}{\omega_{p0}}\right)^2\left[1 - \frac{c^2}{\mu\epsilon}\left(\frac{k}{\omega}\right)^2\right], \tag{3.9}$$

where the function W is defined as

$$W(x, y) = -2\pi\int_0^\infty \frac{df_0}{du}\left\{x\left[\left(1+\frac{u^2}{c^2}\right)^{1/2} - y\right]^2 - \frac{u^2}{x}\right\} \tanh^{-1}\left\{\frac{u}{x}\left[\left(1+\frac{u^2}{c^2}\right)^{1/2} - y\right]^{-1}\right\} du \tag{3.10}$$

and the constant K is defined as

$$K = 2\pi\int_0^\infty f_0\, du. \tag{3.11}$$

In expression (3.10) the argument of $\tanh^{-1}$ indicates that for the fast wave (positive mode)

$$\omega^2 - \omega_b^2 \geqslant k^2c^2 \tag{3.12}$$

while for the slow wave (negative mode)

$$\omega/k \geqslant c. \tag{3.13}$$

In figure 2 the dotted line shows the low-frequency limit of the fast wave. Figures 2 and 3 indicate that all curves vary asymptotically to the values given by

$$\left(\frac{\omega}{\omega_{p0}}\right)^2 = -\frac{4\pi}{3}\int_0^\infty \frac{df_0}{du}u^3\left[\left(1+\frac{u^2}{c^2}\right)^{1/2} \mp \frac{\omega_b}{\omega}\right]^{-1} du. \tag{3.14}$$

In figure 4 a comparison of the asymptotic values of ω/ω_{p0} with the classical results (Stix 1962, pp 27–44) reveals quantitatively the relativistic effect. The curves were plotted in a logarithmic scale so that the percentage differences are clearly visible.

The results presented in this section are essentially in agreement with the previous works (e.g. Buti 1963b, Misra 1975). However, our treatment differs in the following aspects. Firstly, our results are based upon the equilibrium distribution function with no restriction other than isotropy; thus they are free from the problems arising from approximations of f_0 as discussed by Misra (1975). Secondly, we have obtained detailed exact expressions for the general case instead of approximate ones in the limits of weak and strong magnetic fields and in the ultra-relativistic limit.

4. Propagation normal to B_0

When $\theta = \frac{1}{2}\pi$, A_{xz} and A_{yz} also vanish. Thus the dispersion relation is reduced to

$$2\pi\frac{\omega_{p0}^2}{\omega\omega_b}R_{zz} = 1 - \frac{c^2}{\mu\epsilon}\left(\frac{k}{\omega}\right)^2 \tag{4.1}$$

and

$$\left(2\pi\frac{\omega_{p0}^2}{\omega\omega_b}R_{xx} - 1\right)\left[1 - \frac{c^2}{\mu\epsilon}\left(\frac{k}{\omega}\right)^2 - 2\pi\frac{\omega_{p0}^2}{\omega\omega_b}R_{yy}\right] = \left(2\pi\frac{\omega_{p0}^2}{\omega\omega_b}R_{xy}\right)^2. \tag{4.2}$$

Relation (4.1) describes the propagation of uncoupled transverse waves, while relation (4.2) represents the coupled longitudinal and transverse waves.

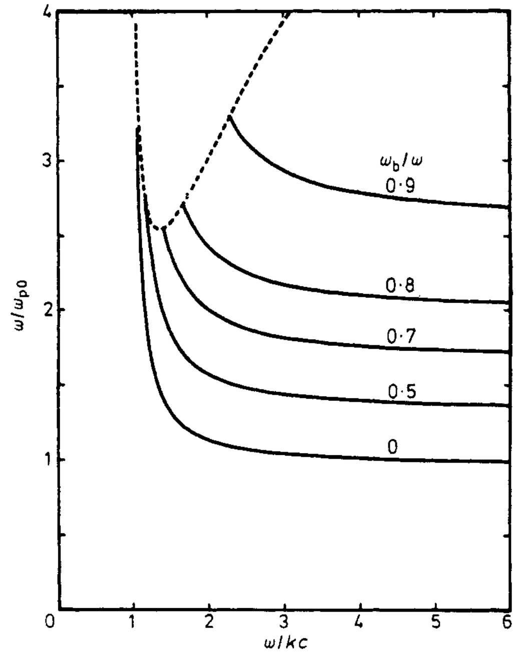

Figure 2. Fast waves (transverse) propagating in a direction parallel to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0c^2/(\kappa T) = 50$.

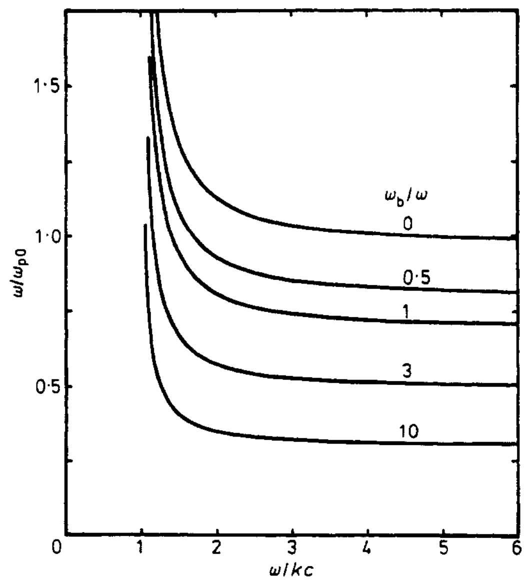

Figure 3. Slow waves (transverse) propagating in a direction parallel to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0c^2/(\kappa T) = 50$.

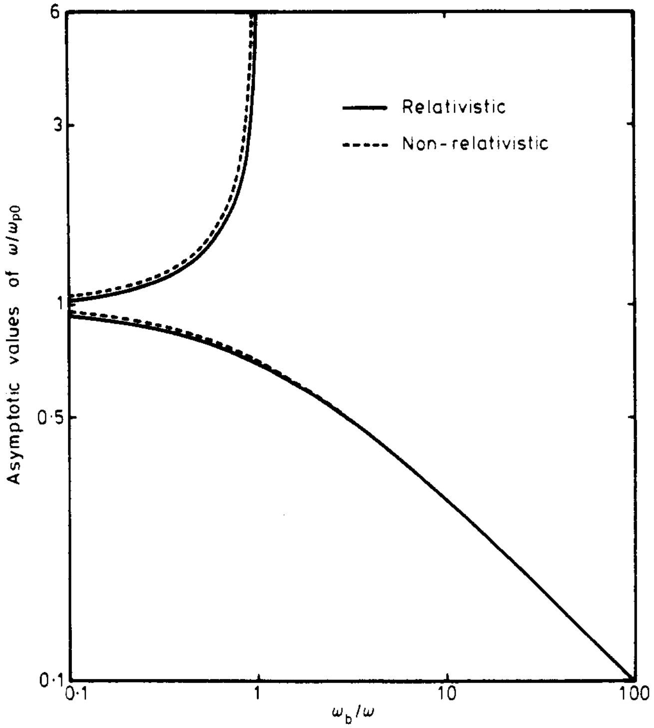

Figure 4. Comparison of asymptotic behaviours with the non-relativistic results for the coupled transverse waves propagating in a direction parallel to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0c^2/(\kappa T) = 50$.

4.1. *Uncoupled transverse mode*

By equations (A.1) and (A.7) in the appendix, after cancellation of equivalent expressions, equation (4.1) may be written as

$$Z_\perp\left(\frac{\omega}{k}, \frac{\omega_b}{\omega}\right) + R_\infty = \left(\frac{\omega}{\omega_{p0}}\right)^2 \left[1 - \frac{c^2}{\mu\epsilon}\left(\frac{k}{\omega}\right)^2\right], \tag{4.3}$$

where the function $Z_\perp$ is defined as

$$\begin{aligned} Z_\perp(x, y) = -4\pi \int_0^\infty \frac{\mathrm{d}f_0}{\mathrm{d}u} u^3 \left(1 + \frac{u^2}{c^2}\right)^{1/2} \sum_{n=1}^\infty \frac{(-1)^n}{(2n+1)(2n+3)} \left(\frac{u}{xy}\right)^{2n} \\ \times \left[(n!)^{-2}\left(1 + \frac{u^2}{c^2}\right)^{-1} + 2 \sum_{r=1}^n \frac{(-1)^r}{(n-r)!(n+r)!} \left(1 + \frac{u^2}{c^2} - r^2 y^2\right)^{-1}\right] \mathrm{d}u. \end{aligned} \tag{4.4}$$

With the aid of the identity

$$(n!)^{-2}(1+\xi^2)^{-1} + 2 \sum_{r=1}^n \frac{(-1)^r}{(n-r)!(n+r)!}(1+\xi^2 - r^2y^2)^{-1} = (-1)^n y^{2n} \Psi_n(\xi, y), \tag{4.5}$$

where Ψ_n denotes a sequence of functions possessing the property

$$\Psi_n(\xi, y) = (1 + \xi^2 - n^2y^2)^{-1} \Psi_{n-1}(\xi, y)$$

with $\Psi_0(\xi, y)=1$, expression (4.4) can be reduced to

$$Z_\perp(x,y)=-4\pi\int_0^\infty \frac{\mathrm{d}f_0}{\mathrm{d}u}u^3\left(1+\frac{u^2}{c^2}\right)^{-1/2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}\left(\frac{u}{x}\right)^{2n}\Psi_n\left(\frac{u}{c},y\right)\mathrm{d}u. \tag{4.6}$$

As shown in figure 5, the numerical results of expression (4.6) indicate that the function $Z_\perp(x, y)$ is practically independent of y. Therefore, equation (4.3) may be reduced to

$$\left(\frac{k}{\omega_{\mathrm{p}0}}\right)^2\left[1-\frac{c^2}{\mu\epsilon}\left(\frac{k}{\omega}\right)^2\right]=M-Z_\parallel\left(\frac{\omega}{k}\right)-2\pi\frac{k}{\omega}\int_0^\infty\frac{\mathrm{d}f_0}{\mathrm{d}u}u^2\tanh^{-1}\left[\frac{ku}{\omega}\left(1+\frac{u^2}{c^2}\right)^{-1/2}\right]\mathrm{d}u. \tag{4.7}$$

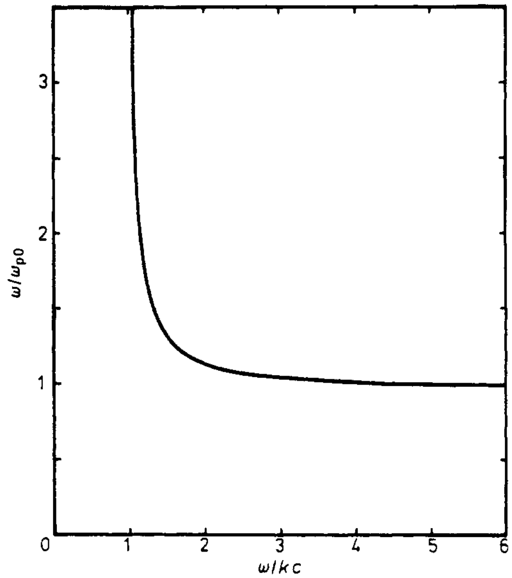

Figure 5. Transverse waves (uncoupled) propagating in a direction normal to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0c^2/(\kappa T)=50$.

Equation (4.7) indicates that the external magnetic field has no effect on the uncoupled transverse wave. Since the right-hand side of equation (4.7) is always positive (see figure 5), it implies

$$\omega/(kc)>(\mu\epsilon)^{-1}>1.$$

4.2. Coupled longitudinal and transverse modes

Phenomena of the coupled longitudinal and transverse waves have so far not been studied by other authors. The solution of equation (4.2) may be expressed as

$$\left(\frac{\omega}{\omega_{\mathrm{p}0}}\right)^2=Q_{xx}+\frac{Q_{yy}}{1-(c^2/\mu\epsilon)(k/\omega)^2}$$
$$\pm\left[\left(Q_{xx}-\frac{Q_{yy}}{1-(c^2/\mu\epsilon)(k/\omega)^2}\right)^2-\frac{4Q_{xy}^2}{1-(c^2/\mu\epsilon)(k/\omega)^2}\right]^{1/2} \tag{4.8}$$

where $\boldsymbol{Q}$, defined as $\pi(\omega/\omega_b)\boldsymbol{R}$ for simplicity, has the elements

$$Q_{xx} = -2\pi \int_0^\infty \frac{df_0}{du} u^3 \left(1+\frac{u^2}{c^2}\right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{2n+1}\left(\frac{ku}{\omega}\right)^{2n-2} \Psi_n\left(\frac{u}{c}, \frac{\omega_b}{\omega}\right) du, \tag{4.9}$$

$$Q_{yy} = -2\pi \int_0^\infty \frac{df_0}{du} u^3 \left(1+\frac{u^2}{c^2}\right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{4n^2-1}\left(\frac{ku}{\omega}\right)^{2n-2} \times\left[1+2n^2(n-1)\left(\frac{\omega_b}{\omega}\right)^2\left(1+\frac{u^2}{c^2}\right)^{-1}\right]\Psi_n\left(\frac{u}{c}, \frac{\omega_b}{\omega}\right) du \tag{4.10}$$

and

$$Q_{xy} = 2\pi i \frac{\omega_b}{\omega} \int_0^\infty \frac{df_0}{du} u^3 \sum_{n=1}^{\infty} \frac{n}{2n+1}\left(\frac{ku}{\omega}\right)^{2n-2} \Psi_n\left(\frac{u}{c}, \frac{\omega_b}{\omega}\right) du. \tag{4.11}$$

Expressions (4.9)–(4.11) were derived from equations (2.10) and (2.11) by use of formulae (A.7)–(A.9) and identities (4.5) and (4.12):

$$\sum_{r=1}^{n} \frac{(-1)^{n-r} r^2}{(n-r)!(n+r)!}(1+\xi^2-r^2y^2)^{-1} = \frac{1}{2} y^{2n-2} \Psi_n(\xi, y). \tag{4.12}$$

It is interesting to note that, by assuming $\omega/k \gg c$ in equation (4.8) and expressions (4.9)–(4.11), the asymptotic limit of frequency ratio ω/ω_{p0} is also given by equation (3.14). Figures 6 and 7 show numerical illustrations of the coupled longitudinal and transverse waves in the limit of real frequency.

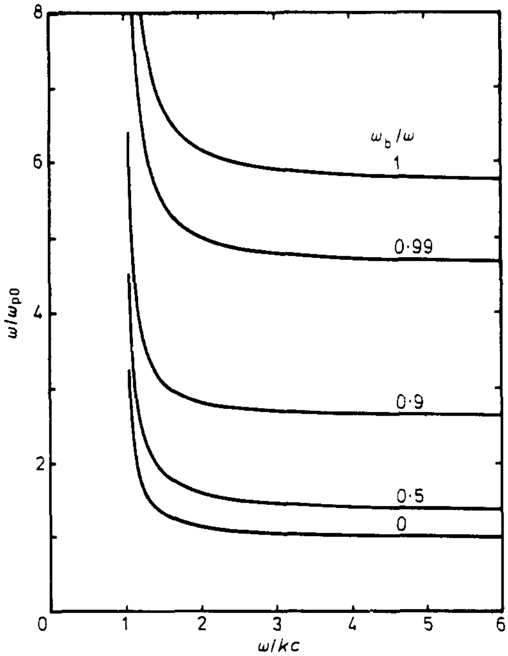

Figure 6. Fast waves (coupled longitudinal and transverse) propagating in a direction normal to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0c^2/(\kappa T) = 50$.

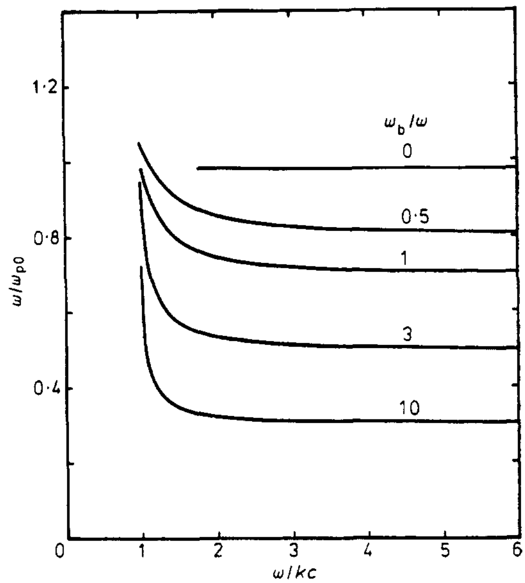

Figure 7. Slow waves (coupled longitudinal and transverse) propagating in a direction normal to $\boldsymbol{B}_0$ for the Maxwellian distribution with $m_0c^2/(\kappa T) = 50$.

5. Conclusions

Propagation of disturbances through a relativistic Vlasov plasma in the presence of a uniform external magnetic field has been investigated. Solutions of the dispersion relation are based on possible evaluations of the dyadic $\boldsymbol{R}$. The exact analytic expressions for an arbitrary isotropic equilibrium distribution function in the form of single integrals have been derived for the propagations parallel and perpendicular to the external magnetic field. The expressions for the limits and asymptotic values of the frequency ratios have also been obtained.

Appendix. Definite integrals involving Bessel functions

We shall derive the expressions for the definite integrals

$$\int_0^{\pi/2} \sin^p \theta J_m(x \sin \theta) J_n(x \sin \theta) \, d\theta$$

$$\int_0^{\pi/2} \sin^p \theta J_m(x \sin \theta) J_n'(x \sin \theta) \, d\theta$$

and

$$\int_0^{\pi/2} \sin^p \theta J_m'(x \sin \theta) J_n'(x \sin \theta) \, d\theta.$$

Let us first define the function

$$\Lambda(n)=\int_0^{\pi/2}\sin^n\theta\,\mathrm{d}\theta=\begin{cases}\dfrac{2\,.\,4\,.\,6\,.\,.\,.\,(n-1)}{1\,.\,3\,.\,5\,.\,.\,.\,n} & \text{for odd } n\\[2ex] \dfrac{\pi}{2}\,\dfrac{1\,.\,3\,.\,5\,.\,.\,.\,(n-1)}{2\,.\,4\,.\,6\,.\,.\,.\,n} & \text{for even } n\end{cases} \tag{A.1}$$

(Dwight 1961).

The series expansion of the Bessel function leads to

$$J_m(x)J_n(x)=\left(\frac{x}{2}\right)^{m+n}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\sum_{p=0}^{r}\frac{1}{(r-p)!(m+r-p)!p!(n+p)!}.$$

By the identity for binomial coefficients (Abramowitz and Stegun 1965, p 822)

$$\sum_{p=0}^{r}\binom{m+r}{p}\binom{n+r}{r-p}=\binom{m+n+2r}{r},$$

we may express

$$J_m(x)J_n(x)=\left(\frac{x}{2}\right)^{m+n}\sum_{r=0}^{\infty}\frac{(-1)^r}{(m+r)!(n+r)!}\binom{m+n+2r}{r}\left(\frac{x}{2}\right)^{2r}. \tag{A.2}$$

Similarly, with the aid of the binomial identities

$$\sum_{p=0}^{r}(n+2p)\binom{m+r}{p}\binom{n+r}{r-p}=(n+r)\binom{m+n+2r}{r}+(m-n)\binom{m+n+2r-1}{r-1} \tag{A.3}$$

and

$$\begin{aligned}\sum_{p=0}^{r}&(m+2r-2p)(n+2p)\binom{m+r}{r}\binom{n+r}{r-p}\\&=(m+r)(n+r)\binom{m+n+2r}{r}+[(m-n)^2-(m+n+2r)]\binom{m+n+2r-2}{r-1}\end{aligned} \tag{A.4}$$

we may express

$$\begin{aligned}J_m(x)J_n'(x)=&\frac{1}{x}\left(\frac{x}{2}\right)^{m+n}\sum_{r=0}^{\infty}\frac{(-1)^r}{(m+r)!(n+r)!}\left(\frac{x}{2}\right)^{2r}\\&\times\left[(n+r)\binom{m+n+2r}{r}+(m-n)\binom{m+n+2r-1}{r-1}\right]\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}J_m'(x)J_n'(x)=&\frac{1}{x^2}\left(\frac{x}{2}\right)^{m+n}\sum_{r=0}^{\infty}\frac{(-1)^r}{(m+r)!(n+r)!}\left(\frac{x}{2}\right)^{2r}\\&\times\left[(m+r)(n+r)\binom{m+n+2r}{r}+[(m-n)^2-(m+n+2r)]\right.\\&\left.\times\binom{m+n+2r-2}{r-1}\right].\end{aligned} \tag{A.6}$$

The proofs of identities (A.3) and (A.4) are straightforward. If we apply the binomial theorem to the expression

$$x^n(1+x^2)^{m+r}(1+y^2)^{n+r}$$

and differentiate with respect to x once, then set $y=x$ and differentiate with respect to x^2 for r times, after putting $x=0$ we obtain identity (A.3). If we apply the binomial theorem to the expression

$$x^n y^m(1+x^2)^{m+r}(1+y^2)^{n+r}$$

and differentiate with respect to x and y each once, then set $y=x$ and differentiate with respect to x^2 for r times, after putting $x=0$ we obtain identity (A.4).

By equations (A.1), (A.2), (A.5), and (A.6), we have

$$\int_0^{\pi/2} \sin^p \theta J_m(x\sin\theta)J_n(x\sin\theta)\,\mathrm{d}\theta = \sum_{r=0}^{\infty}(-1)^r\frac{\Lambda(m+n+p+2r)}{(m+r)!(n+r)!}\binom{m+n+2r}{r}\left(\frac{x}{2}\right)^{m+n+2r}, \tag{A.7}$$

$$\int_0^{\pi/2} \sin^p \theta J_m(x\sin\theta)J_n'(x\sin\theta)\,\mathrm{d}\theta = \frac{1}{x}\sum_{r=0}^{\infty}(-1)^r\frac{\Lambda(m+n+p-1+2r)}{(m+r)!(n+r)!}\left(\frac{x}{2}\right)^{m+n+2r} \times\left[(n+r)\binom{m+n+2r}{r}+(m-n)\binom{m+n+2r-1}{r-1}\right] \tag{A.8}$$

and

$$\int_0^{\pi/2} \sin^p \theta J_m'(x\sin\theta)J_n'(x\sin\theta)\,\mathrm{d}\theta = \frac{1}{x^2}\sum_{r=0}^{\infty}(-1)^r\frac{\Lambda(m+n+p-2+2r)}{(m+r)!(n+r)!}\left(\frac{x}{2}\right)^{m+n+2r} \times\left[(m+r)(n+r)\binom{m+n+2r}{r}+[(m-n)^2-(m+n+2r)] \times\binom{m+n+2r-2}{r-1}\right]. \tag{A.9}$$

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